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Luttinger's Model and the Matter of Dispersion

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In the Luttinger model, right-hand-going particles moving at a constant velocity c in 1D maintain a constant distance apart. Therefore their interaction energy *should* be a trivial constant of the motion whereas, in fact it is not. This interaction affects the dispersion of all the elementary excitations in the model, hence the dynamics and thermodynamics. (Identical remarks hold for the left-hand-goers.) The present paper explores the meaning of this interesting facet of Luttinger's model.

KEY WORDS: Luttinger; dispersion.

Joaquin ("Quin") M. Luttinger was a master at distilling complex problems that are the daily fare of a theoretical physicist, reducing them to conceptually neat formulations that could be used and re-used by others. One of many such examples that have stood the test of time informs this article. Later named the "Luttinger liquid" to distinguish it from Landau's "Fermi liquid," this model of interacting fermions in one spatial dimension (1D) was originally a purely formal construct, an extension of Thirring's toy model[(1)] to the many-body problem. Yet it appears to yield the universal framework (a sort of "law of corresponding states,") for all kinds of systems of interacting particles—not just fermions—in 1D. Its simplicity also allowed it to resolve divers experimental situations in higher-dimensional electron physics, ranging from the solution to the Kondo impurity problem in 3D metal physics, to the classification of the edge states in the 2D quantum Hall effect.

After a brief review of the model and its history, we shall examine a special feature of this model relating to the dispersion of the elementary boson particles, $e(q)$.

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Luttinger's article[2] appeared in print just at the time Elliott H. Lieb and I were planning a book on one-dimensional physics, later published in 1966 by Academic Press under the title *Mathematical Physics in One Dimension*. But before including such a novel theory in our collection we undertook to understand it thoroughly. Our study of what initially seemed a paradoxical semi-relativistic model of interacting spinless electrons in 1D did lead us to construct something new and unexpected: a field-theoretic version of the original theory that could still be solved in closed form without any of the original paradoxes, *via* a transformation later to become known as "bosonization."

Yet it was with some trepidation that I brought our manuscript to Professor Luttinger's office in Columbia University, fearful that some hidden vice might invalidate our version of his model. I need not have worried; within a few hours he had duplicated our calculations and in a characteristically generous manner, telephoned to congratulate Elliott and me. Thus expeditiously refereed, our paper[3] was published shortly thereafter. Ever since that time, literally hundreds of researchers have become equally bemused by the classic simplicity of Luttinger's model, and in thousands of published works have sought to apply it, to extend it and, less successfully, to generalize it.[4]

The present work deals with just one aspect of Luttinger's model as he originally proposed it,[2] *viz.* his artful neglect of any interactions of right-hand-going (*rhg*) particles with one another $H' = \sum_{i \neq i'} U(x_i - x_{i'})$ and, by symmetry, of left-hand-goers (*lhg*) with one another, H'_y. Recall his original Hamiltonian:

$$H = \frac{\hbar c}{i}\left(\sum_i \partial/\partial x_i - \sum_j \partial/\partial y_j\right) + \sum_i \sum_j V(x_i - y_j) \tag{1}$$

All *rhg* particles move at the same speed, $+c$, hence each of $N(N-1)/2$ separations $x_i - x_{i'}$ is a constant of the motion. $H' = H'_x + H'_y$ is itself a constant of the motion, although an uninteresting one at that. (It is simple enough to verify that $[H', H] \equiv 0$.) Too many constants of the motion gives rise to some unease; is the model over- or underconstrained? Are the "constants" not, in fact, variables? Luttinger had intuitively determined that the quickest way to eliminate the ambiguities was to simply choose $U \equiv 0$ in H'.

In what follows I shall show that, in the boson field theory which best expresses the physical content of the Luttinger model, H' does *not* commute with H and therefore there are no extra constants of the motion. The inclusion of H' merely modifies the dispersion of the elementary excitations and as such helps to determine the dynamics and thermodynamics of the model and of its constituent quasiparticles.

For simplicity in the exposition, let us first consider just the *rhg* particles and their mutual interactions, i.e.: $H_x = (\hbar c/i) \sum_i \partial/\partial x_i + \sum_i \sum_{j \neq i} U(x_i - x_j)$. Introducing fermion creation/annihilation operators c^+ and c, this becomes:

$$H_x = \hbar c \sum_{\text{all } k} k c_k^+ c_k + \frac{1}{L} \sum_{q>0} u_q : \rho_q \rho_{-q} : \tag{2}$$

where $: \cdots :$ indicates normal ordering, L is the size of the system, u_q is the Fourier transform of $U(x)$ (assumed real) and $\rho_q = \sum_k c_{k+q}^+ c_k$ is the fermion density operator. A first, nontrivial, step consists of filling the Fermi sea while conserving momentum: $c_k \to a_k$ for $k > 0$, and $c_k \to b_{-k}^+$ for $k < 0$. In addition to some constant "vacuum energy" terms (of little or no interest) this transformation yields the following Hamiltonian governing the dynamical degrees of freedom:

$$H_x = \hbar c \sum_{k>0} k(a_k^+ a_k + b_k^+ b_k) + \frac{1}{L} \sum_{q>0} u_q : \rho_q \rho_{-q} : \tag{3}$$

where now, assuming $q > 0$,

$$\rho_q = \sum_{k=-\infty}^{-q} b_{-(k+q)} b_{-k}^+ + \sum_{k=-q}^{0} a_{k+q}^+ b_{-k}^+ + \sum_{k=0}^{\infty} a_{k+q}^+ a_k, \qquad \text{and}$$

$$\rho_{-q} = \sum_{k=-\infty}^{-q} b_{-k} b_{-(k+q)}^+ + \sum_{k=-q}^{0} b_{-k} a_{k+q} + \sum_{k=0}^{\infty} a_k^+ a_{k+q} \tag{4}$$

We now identify the algebra that enables this model to be brought into diagonal form in terms of its boson-like elementary excitations exclusively, by setting $\rho_q = \sqrt{qL/2\pi}\, \alpha_q^+$ (for $q > 0$) and for its Hermitian conjugate, $\rho_{-q} = \sqrt{qL/2\pi}\, \alpha_q$. By direct calculations using (4) one finds that the only nonvanishing, nontrivial commutator is $[\alpha_q, \alpha_{q'}^+] = \delta_{q,q'}$, hence that the α's are ordinary bosons. By checking that $[\rho_{-q}, \hbar c \sum_{k>0} k(a_k^+ a_k + b_k^+ b_k)] = \hbar c q \rho_{-q}$ we establish that α_q is a lowering operator for the entire motional energy term, i.e., that the operator in Eq. (3) can be expressed in terms of these bosons alone.

$$H_x = \sum_{q>0} e(q)\, \alpha_q^+ \alpha_q, \qquad \text{with} \quad e(q) = \hbar c\, |q|\, (1 + 2\pi u_q/\hbar c) \tag{5}$$

For the *lhg* particles, a similar procedure yields H_y virtually identical to the above, except that the index $q<0$. Without loss of generality we assume $u_{-q}=u_{+q}$, therefore $e(-q)=e(q)$. Combining the two, introducing v_q (also taken to be real) as the Fourier transform of $V(x-y)$ and again omitting irrelevant constants, we obtain for the full Hamiltonian, including interactions, i.e., the reformulated Eq. (1) cum H':

$$H=\sum_{\text{all } q} e(q)\,\alpha_q^+\alpha_q+2\pi\sum_{q>0} qv_q(\alpha_q^+\alpha_{-q}^+ +\alpha_{-q}\alpha_q) \tag{6}$$

The final diagonalization of H is carried out separately in each q-sector for essentially arbitrary $e(q)$ and v_q. After an appropriate unitary transformation the final result is,

$$H=\sum_{\textit{all } q}\omega(q)\,\alpha_q^+\alpha_q, \qquad \text{where} \quad \omega(q)=\hbar c\,|q|\,\sqrt{(1+2\pi u_q/\hbar c)^2-(2\pi v_q/\hbar c)^2} \tag{7}$$

to within some additive constants. Which finally brings us to the point of this paper, the nature of $e(q)$ and of $\omega(q)$.

The Luttinger choice is $U=0$, hence $e=\hbar c\,|q|$. If instead one chooses the delta function potential $U(x)=g\delta(x)$ the result is $e=\hbar c\,|q|\,(1+2\pi g/\hbar c)$, a renormalization of c. *All other* choices lead to nontrivial dispersion $e(q)$ for the *rhg* excitations, prior to their scattering off the *lhg* particles.

However, for identical particles there is only one "natural" choice: $U=V$. Hence by Eq. (7),

$$\omega(q)=\hbar c\,|q|\,\sqrt{1+4\pi u_q/\hbar c} \tag{8}$$

This indicates an instability (absence of a ground state) only in the case of excessively attractive potentials. For a plasma, defined by the repulsive Coulomb potential $u_q=v_q\propto e^2/q^2$, Eqs. (7) and (8) result in a stable spectrum with a gap. (By contrast, if we set $U=0$ we would find ω becoming imaginary for *any* strong potential $V(x-y)$—whether attractive *or* repulsive—including the aforementioned Coulomb potential and regardless of its sign.)

In conclusion, the potential energy of a train of particles moving in the same direction at a uniform speed would *seem* to be a constant dictated by initial conditions and incapable of affecting the speed of the particles. Yet in the bosonized version of Luttinger's model,[3,4] this interaction energy, in conjunction with the uncertainty principle and certain other mysterious features of quantum field theory, does significantly affect both the speed and the dispersion of the train of particles $e(p)$ and ultimately the spectrum of the elementary excitations $\omega(p)$ as well.

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